THE STRONG TREE PROPERTY AND THE FAILURE OF SCH

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Abstract. Fontanella [2] showed that if $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of supercompacts and $\nu = \sup_n \kappa_n$, then the strong tree property holds at ν^+. Building on a proof by Neeman [7], we show that the strong tree property at κ^+ is consistent with $\neg SCH_\kappa$, where κ is singular strong limit of countable cofinality.

1. Introduction

The tree property at a regular cardinal κ, denoted TP_κ, states that every κ-tree of height κ and levels of size $< \kappa$ has an unbounded branch. When κ is inaccesible, TP_κ is equivalent to weak compactness of κ. A major open project in set theory is obtaining the tree property at as many small regular cardinals as possible. This tests how much compactness a universe of set theory can have. Classical results by König[4] and Aronszajn[6] respectively show that $TP_{\aleph_0}$ holds and $TP_{\aleph_1}$ fails. Specker[12] generalized Aronszajn's result to show that if $\kappa^{<\kappa} = \kappa$, then TP_{κ^+} fails. In particular, if κ is singular strong limit, then $TP_{\kappa^{++}}$ requires $\neg SCH_\kappa$.

A major open project is consistently obtaining the tree property at many small cardinals. This tests the power of forcing and large cardinals to build universes with a high level of compactness. One of the best known results is due to Neeman[8], who showed that $TP_{\aleph_\alpha}$ can consistently hold for $\alpha \in [2, \omega) \cup \{\omega + 1\}$ simultaneously with $\aleph_\omega$ strong limit. An open question is whether this can be extended to include $TP_{\aleph_{\omega+2}}$. By Specker's Theorem, a necessary condition would be obtaining the tree property at $\aleph_{\omega+1}$ with $\neg SCH_{\aleph_\omega}$. That problem is also open. However, Sinapova and Unger[11] have shown that the tree property can consistently hold at the successor and double successor of a singular strong limit cardinal.

Just as the tree property captures the "essence" of weakly compact cardinals, the strong and super tree properties respectively capture the "essence" of strongly and super compact cardinals: when κ is inaccessible, κ is strongly compact iff it has the strong tree property, and supercompact iff it has the super tree property. The strong and super tree properties are defined in terms of $P_\kappa(\lambda)$-lists, which were first studied by Jech and Magidor, and later by Weiss[14]. Fontanella[2] found equivalent characterizations in terms of (κ, λ)-trees. This paper will use her characterization of the strong tree property.

In this paper, $f|X$ will denote the restriction of the function f to the set X, where $\mathrm{dom}(f) \supset X$.

Definition 1. *A (κ, λ)-tree is a set F satisfying the following:*

1. *For every $f \in F$, $f : X \to 2$ for some $X \in [\lambda]^{<\kappa}$*
2. *For all $f \in F$, if $X \subset dom(f)$, then $f|X \in F$*
3. *For all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F : dom(f) = X\} \neq \emptyset$*
4. *For all $X \in [\lambda]^{<\kappa}$, $|Lev_X(F)| < \kappa$*

An unbounded branch through F is a function $b : \lambda \to 2$ such that for every $X \in [\lambda]^{<\kappa}$, $b|X \in Lev_X(F)$.

$TP(\kappa, \lambda)$ holds if every (κ, λ)-tree has an unbounded branch. The strong tree property holds at κ if $TP(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$.

Even more challenging than the project of obtaining the tree proparty at many small regular cardinals is obtaining the strong tree property at many small regular cardinals. Fontanella[3] and Unger[13] independently showed that a model due to Cummings and Foreman[1] has the super tree property at $\aleph_n$ for $n \geq 2$. Fontanella[2] showed that the strong tree property can consistently hold at $\aleph_{\omega+1}$. In the setting of $\aleph_\omega$ strong limit, a good improvement on this result would be consistently obtaining the strong tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$. As in the tree property case, a necessary requirement would be obtaining the strong tree property at $\aleph_{\omega+1}$ with $\neg SCH_{\aleph_\omega}$.

In this paper, building on an argument due to Neeman[7], we answer the question just stated when $\aleph_\omega$ is replaced by some singular strong limit cardinal of countable cofinality. Specifically, we prove the following:

Theorem 1. *Assuming the consistency of ω-many supercompacts $\langle \kappa_n : n < \omega \rangle$, if $\nu = \sup_n \kappa_n$, then there is a model where κ_0 is strong limit, $\neg SCH_{\kappa_0}$ and the Strong Tree Property holds at κ_0^+.*

2. The Forcing

Start with $V_0 \vDash GCH$, $\langle \kappa_n : n < \omega \rangle$ supercompacts and $\nu = \sup_n \kappa_n$. Our construction will be as follows:

- Perform Laver preparation to make κ_0 indestructible with respect to $\mathbb{A} = \mathrm{Add}(\kappa_0, \nu^{++})$; call this model V.
- Force with $\mathbb{A}$. Let E be generic for $\mathbb{A}$ over V; call the resulting model $V[E]$.

In $V[E]$, we use the following poset due to Neeman[7]: let U be a normal measure on $P_{\kappa_0}(\nu^+)$ and U_n be the projection of U on to $P_{\kappa_0}(\kappa_n)$. For sets of ordinals x and y, write $x \prec y$ if $x \subset y$ and $ot(x) < \kappa_0 \cap y$, where $ot(x)$

is the order type of x. Let $\mathbb{P}$ be the following variant of Gitik-Sharon[5] forcing: conditions are of the form $\langle x_0, ..., x_{n-1}, A_n, A_{n+1}, ...\rangle$ where

- $x_i \in P_{\kappa_0}(\kappa_i)$, $x_i \prec x_{i+1}$
- $A_i \in U_i$ and $x_{n-1} \prec y$ for all $y \in A_n$

We require that $\kappa_0 \cap x_i$ is inaccessible.

Given a condition $p = \langle x_0, ..., x_{n-1}, A_n, A_{n+1}, ...\rangle$, let stem$(p) = \langle x_0, ..., x_{n-1}\rangle$. If h is a stem, $\varphi(x_1, ..., x_m)$ is a formula and $a_1, ...a_m$ are parameters, we write $h \Vdash^* \varphi(\dot{a_1}, ..., \dot{a_m})$ if there is a condition p with stem$(p) = h$ such that $p \Vdash \varphi(\dot{a_1}, ..., \dot{a_m})$.

If $p = \langle x_0^p, ...x_{n-1}^p, A_n^p, ...\rangle$ and $q = \langle x_0^q, ...x_{m-1}^q, A_m^q, ...\rangle$, we say $p \leq q$ if $m \leq n$, $x_i^p = x_i^q$ for $i < m$, $x_i^p \in A_i^q$ for $m \leq i < n$ and $A_i^p \subset A_i^q$ for $i \geq n$. As noted by Neeman, $\mathbb{P}$ satisfies the Prikry Property.

Let G be $\mathbb{P}$-generic over $V[E]$. Then in $V[E][G]$, every κ_n is collapsed to κ_0 and $(\nu^+)^{V[E]}$ is the new successor of κ_0. We will write ν^+ for $(\nu^+)^{V[E]}$ and likewise for ν^{++}.

As noted by Neeman[7], $V[E][G] \vDash 2^{\kappa_0} = \kappa_0^{++}$, κ_0 strong limit.

Our main task will be to show that $V[E][G] \vDash TP(\nu^+, \lambda)$ for all $\lambda \geq \nu^+$. Since it is enough to do this for unboundedly many λ, we may assume $\lambda^\nu = \lambda$. The argument closely follows Neeman's[7].

3. The Strong Tree Property at ν^+

Let $F \in V[E][G]$ be a (ν^+, λ)-tree and for each $X \in [\lambda]^{<\nu^+}$ let $\{f_i^X\}_{i<|Lev_X(F)|}$ be an enumeration of $Lev_X(F)$ with $|Lev_X(F)| \leq \kappa$. Let $j : V[E] \to M$ be a λ-supercompact embedding with critical point κ_0.

We point out that $([\lambda]^{<\nu^+})^{V[E]} \neq ([\lambda]^{<\nu^+})^{V[E][G]}$. However, $([\lambda]^{<\nu^+})^{V[E]} = ([\lambda]^{<\nu^+})^M$ because M contains all $V[E]$-sets of size $\leq \lambda$.

Club subsets of $[\lambda]^{<\nu^+}$ in $V[E][G]$ satisfy the following covering property:

Lemma 1. *Let $q \Vdash \dot{K} \subset [\lambda]^{<\nu^+}$ is a club. Then there is a club $C \in V[E]$ such that $q \Vdash C \subset \dot{K}$.*

Proof. Let $C = \{X \in [\lambda]^{<\nu^+} : q \Vdash X \in \dot{K}\}$. Clearly $C \in V[E]$ and $V[E][G] \vDash C \subset \dot{K}_G$. It remains to show that C is club. If $\tau < \nu^+$ and $\langle X_\alpha : \alpha < \tau\rangle$ is an increasing sequence of elements of C, then since $q \Vdash \dot{K}$ is club and $q \Vdash X_\alpha \in \dot{K}$ for all α, $q \Vdash \bigcup_\alpha X_\alpha \in \dot{K}$, i.e. $\bigcup_\alpha X_\alpha \in C$. So C is closed. Let $X_0 \in [\lambda]^{<\nu^+}$ be arbitrary. We will construct an increasing sequence $\langle X_n : n < \omega\rangle$ such that $X = \bigcup_n X_n \in C$. Assuming that we have constructed X_n, let $A \subset \{p \leq q : \exists X \supsetneq X_n(p \Vdash X \in \dot{K})\}$ be an antichain maximal with respect to all such antichains.

Let G' be a generic filter containing q. Since $q \Vdash \dot{K}$ is unbounded, $G' \cap A \neq \emptyset$. Since $\mathbb{P}$ has the ν^+-chain condition, $|A| \leq \nu$. For each $p \in A$, let X_p be such that $p \Vdash X_p \in \dot{K}$. Then $X_{n+1} = \bigcup_{p \in A} X_p \supset X_n \in [\lambda]^{<\nu^+}$.

Now by construction, for any generic G', there is $\langle Y_n : n < \omega \rangle$ with $X_n \subset Y_n \subset X_{n+1}$ such that $V[E][G'] \vDash Y_n \in \dot{K}_{G'}$. Since $\bigcup_n Y_n = \bigcup_n X_n = X$, $V[E][G'] \vDash X \in \dot{K}_{G'}$. Finally, since G' was arbitrary, $q \Vdash X \in \dot{K}$. So C is unbounded. $\square$

As an immediate consequence, we see that $([\lambda]^{<\nu^+})^{V[E]}$ is stationary in $([\lambda]^{<\nu^+})^{V[E][G]}$. From now on, $[\lambda]^{<\nu^+}$ will mean $([\lambda]^{<\nu^+})^{V[E]}$ unless otherwise specified.

We will also need the following approximation property.

Definition 2. *Let G be generic for $\mathbb{P}$ over V and κ be a cardinal in $V[G]$. We say that $\mathbb{P}$ has the κ-approximation property if for every $A \in V[G]$ such that $A \cap D \in V$ for every $|D| < \kappa$, $A \in V$.*

Claim 2. *$j(\mathbb{A})/E$ has the ν^+-approximation property.*

Proof. $j(\mathbb{A})$ is κ_0^+-Knaster, hence $j(\mathbb{A}) \times j(\mathbb{A})$ has the κ_0^+-c.c. Then $j(\mathbb{A})/E \times j(\mathbb{A})/E$ also has the κ_0^+-c.c., and in particular the ν^+-c.c. By a lemma due to Unger[13], $j(\mathbb{A})/E$ has the ν^+-approximation property. $\square$

Lemma 3. *$\exists n \exists S \subset [\lambda]^{<\nu^+}$ stationary in $V[E]$ such that for all $X, Y \in S$, $\exists \zeta, \eta < \kappa_0 \exists p \in \mathbb{P}$ with length$(p) = n$ such that $p \Vdash \dot{f}_\zeta^X | (X \cap Y) = \dot{f}_\eta^Y | (X \cap Y)$.*

Proof. $j(\dot{F})$ is a $j(\mathbb{P})$-name for a $(j(\nu^+), j(\lambda))$-tree. Let G^* be M-generic for $j(\mathbb{P})$. Then $j(\dot{F})_{G^*}$ is a $(j(\nu^+), j(\lambda))$-tree. Write f_i^{*X} for the i^{th} node on the X^{th} level of $j(\dot{F})_{G^*}$.

Let $Z = \bigcup \{j(X) : X \in [\lambda]^{<\nu^+}\}$. $Z \in M$ because $\lambda^\nu = \lambda$ and M is closed under λ-sequences. Since the size of each $j(X)$ is less than $j(\nu^+)$, $|Z| \leq j(\nu) \cdot \lambda = j(\nu)$. Furthermore, $M \subset M[G^*]$. So $Z \in ([j(\lambda)]^{<j(\nu^+)})^{M[G^*]}$. Take u a node on the Z^{th} level of $j(\dot{F})_{G^*}$.

For each $X \in [\lambda]^{<\nu^+}$, $Z \supset j(X)$, so in $M[G^*]$, $u | j(X)$ is a node on the $j(X)^{\text{th}}$ level. Let $p_X \in G^*$ be such that $p_X \Vdash \dot{u} | j(X) = \dot{f}_{\zeta_X}^{*j(X)}$ for some $\zeta_X < j(\kappa_0)$ and $n_X = \text{length}(p_X)$. The function $X \mapsto n_X$ ($X \in [\lambda]^{<\nu^+}$) can be defined in $M[G^*]$, it's domain is a stationary subset of $([\lambda]^{<\nu^+})^{M[G^*]}$. Since ν^+ remains regular in $M[G^*]$, we may find stationary $S^* \subset [\lambda]^{<\nu^+}$ in $M[G^*]$ such that $n_X = n$ on S^* for some constant n. Compatible conditions of the same length must have the same stem; so let h be the common stem of all p_X such that $X \in S^*$.

Define in M, $S = \{X \in [\lambda]^{<\nu^+} : \exists p \in j(\mathbb{P})(\text{stem}(p) = h \wedge \exists \zeta < j(\kappa_0)(p \Vdash \dot{u}|j(X) = \dot{f}^{*j(X)}_{\zeta}))\}$. Clearly, $S \supset S^*$ as witnessed by p_X for each $X \in S$. So S is stationary. If $X, Y \in S$ as witnessed by p_X, ζ_X and p_Y, ζ_Y respectively, then $p_X \wedge p_Y$ forces $\dot{f}^{*j(X)}_{\zeta_X}$ and $\dot{f}^{*j(Y)}_{\zeta_Y}$ to be restrictions of $\dot{u}$, hence $p_X \wedge p_Y \Vdash \dot{f}^{*j(X)}_{\zeta_X}|j(X) \cap j(Y) = \dot{f}^{*j(Y)}_{\zeta_Y}|j(X) \cap j(Y)$. Note that $\text{length}(p_X \wedge p_Y) = n$.

Now for any $X, Y \in S$, we have:
$M \models \exists \zeta, \eta < j(\kappa_0) \exists p \in j(\mathbb{P})(\text{length}(p) = n \wedge p \Vdash \dot{f}^{*j(X)}_{\zeta}|j(X) \cap j(Y) = \dot{f}^{*j(Y)}_{\eta}|j(X) \cap j(Y))$

By elementarity:
$V[E] \models \exists \zeta, \eta < \kappa_0 \exists p \in \mathbb{P}(\text{length}(p) = n \wedge p \Vdash \dot{f}^X_{\zeta}|(X \cap Y) = \dot{f}^Y_{\eta}|(X \cap Y))$ $\square$

Let n be as in Lemma 3. Let $j_1 : V \to N$ be a λ-supercompactness embedding with $\text{crit}(j_1) = \kappa_{n+1}$. Then $j_1(\mathbb{A}) = \text{Add}(\kappa_0, j_1(\nu^{++}))$. Let E' be generic for $j_1(\mathbb{A})$ over N containing j_1"E. We can then lift j_1 to an embedding from $V[E]$ to $N[E']$, which we will continue to denote by j_1.

Lemma 4. *$\exists T \subset [\lambda]^{<\nu^+}$ stationary in $V[E]$, a stem $\bar{h}$ of length n and for each $X \in T$ an ordinal $\zeta_X < \kappa_0$ such that for all $X, Y \in T$, there is p with stem $\bar{h}$ such that $p \Vdash \dot{f}^X_{\zeta_X}|(X \cap Y) = \dot{f}^Y_{\zeta_Y}|(X \cap Y)$.*

Proof. Proceeding as in the proof of Lemma 3, let $Z' = \bigcup\{j_1(X) : X \in [\lambda]^{<\nu^+}\} \in N[E']$. Since $j_1(S)$ is stationary in $[j_1(\lambda)]^{<j_1(\nu^+)}$, it is unbounded, so let $Z \in j_1(S)$ with $Z \supset Z'$. By Lemma 3 and elementarity of j_1, in N we find for all $X^*, Y^* \in j_1(S)$, $\exists \zeta, \eta < \kappa_0$ and $p \in j_1(\mathbb{P})$ with $\text{length}(p) = n$ such that $p \Vdash j_1(\dot{f})^{X^*}_{\zeta}|(X^* \cap Y^*) = j_1(\dot{f})^{Y^*}_{\eta}|(X^* \cap Y^*)$. In particular, for any $X \in S$, taking $X^* = j_1(X)$, $Y^* = Z$ and noting that $Z \supset j_1(X)$, we can find $p_X \in j_1(\mathbb{P})$ of length n and $\zeta_X, \eta_X < \kappa_0$ such that $p_X \Vdash j_1(\dot{f}^X_{\zeta_X}) = j_1(\dot{f})^Z_{\eta_X}|j_1(X)$. Let h_X be the stem of p_X. Then $X \mapsto \langle h_X, \eta_X \rangle$ is a map from a set of size λ (namely S), to a set of size κ_n (namely $\{s : s \text{ is a stem of length } n\} \times \kappa_0$). Let $T \subset S$ be stationary on which this map is constant. Letting $\langle h, \bar{\eta} \rangle$ be the constant, we have for any $X, Y \in T$, $p_X \wedge p_Y \Vdash j_1(\dot{f}^X_{\zeta_X})|(j_1(X) \cap j_1(Y)) = j_1(\dot{f})^Z_{\bar{\eta}}|(j_1(X) \cap j_1(Y)) = j_1(\dot{f}^Y_{\zeta_Y})|(j_1(X) \cap j_1(Y))$. It follows that for any $X, Y \in T$,
$N[E'] \models \exists p \in j_1(\mathbb{P})(\text{stem}(p) = h \wedge p \Vdash j_1(\dot{f}^X_{\zeta_X})|(j_1(X) \cap j_1(Y)) = j_1(\dot{f}^Y_{\zeta_Y})|(j_1(X) \cap j_1(Y)))$
By elementarity, and noting that h, ζ_X, ζ_Y are below the critical point of j_1,
$V[E] \models \exists p \in \mathbb{P}(\text{stem}(p) = h \wedge p \Vdash \dot{f}^X_{\zeta_X}|(X \cap Y) = \dot{f}^Y_{\zeta_Y}|(X \cap Y))$
Note that $T \in V[E][E']$ because j_1 is defined in $V[E][E']$. However, since $T \subset S$, every $X \in T$ is in $V[E]$. To complete the proof of the lemma, we must

show that $T \in V[E]$. By Claim 2, it is enough to show that $T \cap D \in V[E]$ whenever $D \subset [\lambda]^{<\nu^+}$, $|D| < \nu^+$.

Let D be as above. Since $\bigcup D \in [\lambda]^{<\nu^+}$ and T is unbounded, let $Y \in T$ with $Y \supset \bigcup D$. Let $X \in S \cap D$. From the above, we see that if $X \in T$, then $(\exists\zeta) h \Vdash^* \dot{f}^X_\zeta = \dot{f}^Y_{\zeta_Y}|X$. Conversely, suppose $(\exists\zeta) h \Vdash^* \dot{f}^X_\zeta = \dot{f}^Y_{\zeta_Y}|X$. By elementarity, $N[E'] \vDash h \Vdash^* j_1(\dot{f}^X_\zeta) = j_1(\dot{f}^Y_{\zeta_Y})|j_1(X)$ for some ζ. From the above, we have $h \Vdash^* j_1(\dot{f}^Y_{\zeta_Y}) = j_1(\dot{f})^Z_{\bar\eta}|j_1(Y)$. Hence $h \Vdash^* j_1(\dot{f}^X_\zeta) = j_1(\dot{f}^Y_{\zeta_Y})|j_1(X) = j_1(\dot{f})^Z_{\bar\eta}|j_1(X)$. So $\zeta_X = \zeta$, $h_X = h$ and $\eta_X = \bar\eta$. It follows that $X \in T$.

In conclusion, $T \cap D = \{X \in S \cap D : (\exists\zeta) h \Vdash^* \dot{f}^X_\zeta = \dot{f}^Y_{\zeta_Y}|X\} \in V[E]$. $\square$

From now on, we may assume the function $X \mapsto \zeta_X$ $(X \in T)$ is in $V[E]$. Call this function g.

Lemma 5. *Let $h \supset \bar{h}$ have length k and $T^h \subset T$ be stationary in $V[E]$ such that $\forall X, Y \in T^h, h \Vdash^* \dot{f}^X_{\zeta_X}|(X \cap Y) = \dot{f}^Y_{\zeta_Y}|(X \cap Y)$. Then there is a club C_h and $u_h : C_h \cap T^h \to U_k$ such that whenever $X, Y \in T^h \cap C_h$ and $x \in u_h(X) \cap u_h(Y)$, $h^\wedge x \Vdash^* \dot{f}^X_{\zeta_X}|(X \cap Y) = \dot{f}^Y_{\zeta_Y}|(X \cap Y)$.*

Proof. Let $j_2 : V \to N'$ be as before Lemma 4, except $\mathrm{crit}(j_2) = \kappa_{k+1}$ and $\pi : V[E] \to N'[E'']$ be a lift. Let $Z \in \pi(T^h)$ with $Z \supset \{\pi(X) : X \in [\lambda]^{<\nu^+}\}$ and $\xi = \pi(g)_Z$. Then $Z \cap \pi(X) = \pi(X)$ for any $X \in [\lambda]^{<\nu^+}$.

Claim 6. *There is $v : T^h \to \pi(U_k)$ in $V[E][E'']$ such that for all $X \in T^h$ and $x \in v(X)$, $h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)$.*

Proof. By elementarity of π and noting that $\pi(h) = h$, for all $X^*, Y^* \in \pi(T^h)$, $h \Vdash^* \pi(\dot{f})^{X^*}_{\pi(g)_{X^*}}|(X^* \cap Y^*) = \pi(\dot{f})^{Y^*}_{\pi(g)_{Y^*}}|(X^* \cap Y^*)$. Let $X^* = \pi(X)$ and $Y^* = Z$. Then we have a condition $r_X \in \pi(\mathbb{P})$ with stem h such that $r_X \Vdash \pi(\dot{f})^{\pi(X)}_{\pi(g)_{\pi(X)}} = \pi(\dot{f})^Z_{\pi(g)_Z}|\pi(X)$. Let $v(X) = A^{r_X}_k$. Then for any $x \in v(X)$, we can choose a condition with stem $h \frown x$ extending r_X. Since $\pi(\dot{f})^{\pi(X)}_{\pi(g)_{\pi(X)}} = \pi(\dot{f}^X_{\zeta_X})$, this condition will witness $h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)$. $\square$

For each $x \in P_{\kappa_0}(\kappa_k)$, let $T_x = \{X \in T^h : h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)\} \in V[E][E'']$.

Claim 7. *If T_x is unbounded, then it is in $V[E]$.*

Proof. Let $D = \{X_i : i < \tau\}$ with $\tau < \nu^+$. Since T_x is stationary, and in particular unbounded, there is $X \in T_x$ such that $X \supset \bigcup_{i<\tau} X_i$. By definition, $h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)$. For each $i < \tau$, we then get

$h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X})|\pi(X_i) = \pi(\dot{f})^Z_\xi|\pi(X_i)$.
$X_i \in T_x \iff h \frown x \Vdash^* \pi(\dot{f}^{X_i}_{\zeta_{X_i}}) = \pi(\dot{f})^Z_\xi|\pi(X_i)$
$\iff h \frown x \Vdash^* \pi(\dot{f}^{X_i}_{\zeta_{X_i}}) = \pi(\dot{f}^X_{\zeta_X})|\pi(X_i)$
$\iff h \frown x \Vdash^* \dot{f}^{X_i}_{\zeta_{X_i}} = \dot{f}^X_{\zeta_X}|X_i$
So $T_x|D = \{X_i : h \frown x \Vdash^* \dot{f}^{X_i}_{\zeta_{X_i}} = \dot{f}^X_{\zeta_X}|X_i\} \in V[E]$. By the approximation property, it follows that $T_x \in V[E]$. □

Claim 8. *If $X, Y \in T_x$, then $h \frown x \Vdash^* \dot{f}^X_{\zeta_X}|X \cap Y = \dot{f}^Y_{\zeta_Y}|X \cap Y$.*

Proof. We have $h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)$ and $h \frown x \Vdash^* \pi(\dot{f}^Y_{\zeta_Y}) = \pi(\dot{f})^Z_\xi|\pi(Y)$. So $h \frown x \Vdash^* \pi(\dot{f})^Z_\xi|\pi(X) \cap \pi(Y) = \pi(\dot{f}^X_{\zeta_X})|\pi(X) \cap \pi(Y) = \pi(\dot{f}^Y_{\zeta_Y})|\pi(X) \cap \pi(Y)$. By elementarity, $h \frown x \Vdash^* \dot{f}^X_{\zeta_X}|X \cap Y = \dot{f}^Y_{\zeta_Y}|X \cap Y$. □

In $V[E]$, define $K_x = \{C \subset T^h : C$ is stationary$\wedge \exists q \in \mathrm{Add}(\kappa_0, j_2(\lambda^+))(q \Vdash \dot{T}_x = C)\}$. Then for any $C \in K_x$, C can be T_x so by Claim 8, if $X, Y \in C$, then $h \frown x \Vdash^* \dot{f}^X_{\zeta_X}|X \cap Y = \dot{f}^Y_{\zeta_Y}|X \cap Y$. Furthermore, suppose $C, C' \in K_x$ and $C \neq C'$. Let X belong to one but not the other. For any $Y \supset X$, if $Y \in C \cap C'$, then $X \in C \cap C'$, which is impossible. So C and C' are disjoint on $\{Y : Y \supset X\}$.

For each $x \in P_{\kappa_0}(\kappa_k)$ and distinct $C, C' \in K_x$, let $X_{x,C,C'} \in [\lambda]^{<\nu^+}$ be such that C and C' are disjoint above $X_{x,C,C'}$. There are only κ_k-many such x and $|K_x| \leq \kappa_0$ since $\mathrm{Add}(\kappa_0, j(\lambda^+))$ has the κ_0^+-c.c. So $X^h := \bigcup_{x,C,C'} X_{x,C,C'} \in [\lambda]^{<\nu^+}$. Now for every $x \in P_{\kappa_0}(\kappa_k)$ and every $X \supset X^h$ in $[\lambda]^{<\nu^+}$, there is at most one $C \in K_x$ such that $X \in C$. Let $f(x, X)$ be the unique $C \in K_x$ such that $X \in C$ if it exists, leaving $f(x, X)$ undefined otherwise. Note that $f \in V[E]$.

Claim 9. *Let $X \supset X^h$ be in $[\lambda]^{<\nu^+}$. Then $\{x \in P_{\kappa_0}(\kappa_k) : f(x, X)$ is defined$\} \in U_k$.*

Proof. Towards a contradiction, suppose $\tilde{X} = \{x \in P_{\kappa_0}(\kappa_k) : f(x, X)$ is undefined$\} \in U_k$. Recall from Claim 6 that for every $Y \in T^h$, there is $v(Y) \in \pi(U_k)$ such that whenever $x \in v(Y)$, $h \frown x \Vdash^* \pi(\dot{f}^Y_{\zeta_Y}) = \pi(\dot{f})^Z_\xi|\pi(Y)$. Since $\tilde{X}$ is below $\mathrm{crit}(\pi)$, $\tilde{X} \in \pi(U_k) \iff \pi(\tilde{X}) \in \pi(U_k) \iff \tilde{X} \in U_k$, which we have assumed is the case. For each $Y \in T^h$, $v(X) \cap v(Y) \cap \tilde{X} \in \pi(U_k)$, so let x_Y be in this intersection. There are only κ_k-many possible values of x_Y and stationarily many choices for Y. So there must be $\tilde{T}^h \subset T^h$ stationary such that $x_Y = x$ for $Y \in \tilde{T}^h$ for some x.

Since $x \in v(X)$, $h \frown x \Vdash^* \pi(\dot{f}^X_{\zeta_X}) = \pi(\dot{f})^Z_\xi|\pi(X)$. By definition, $X \in T_x$. Similarly, for every $Y \in \tilde{T}^h$, $x \in v(Y)$ and hence $Y \in T_x$. So T_x contains a stationary set, hence is stationary. By Claim 7, $T_x \in V[E]$. But then $T_x \in K_x$ (as witnessed by the empty condition). On one hand, since $x \in \tilde{X}$, $f(x, X)$ is undefined. On the other hand, $X \in T_x$ and $T_x \in K_x$, so $f(x, X) = T_x$. This is a contradiction. $\square$

Claim 10. *Let $X, X' \supset X^h$ be in $[\lambda]^{<\nu^+}$. Then $\{x \in P_{\kappa_0}(\kappa_k) : f(x, X) = f(x, X')\} \in U_k$.*

Proof. Towards a contradiction, suppose $\tilde{X} = \{x \in P_{\kappa_0}(\kappa_k) : f(x, X) \neq f(x, X')\} \in U_k$. We use the same argument, except we take for each $Y \in T^h$, $x_Y \in v(X) \cap v(X') \cap v(Y) \cap \tilde{X}$. This time we get $X, X' \in T_x$ while $T_x \in K_x$. On one hand, since $x \in \tilde{X}$, $f(x, X) \neq f(x, X')$. On the other hand, $X, X' \in T_x \wedge T_x \in K_x \Rightarrow f(x, X) = f(x, X') = T_x$, which is a contradiction. $\square$

We are finally ready to finish the proof of Lemma 5. Let $X_0 \supset X^h$ be in T^h. Our club will be $C_h = \{X \in [\lambda]^{<\nu^+} : X \supset X^h\}$. For $X \in C_h \cap T^h$, we will use $u_h(X) = A^h_X := \{x \in P_{\kappa_0}(\kappa_k) : f(x, X_0) \text{ is defined} \wedge f(x, X) = f(x, X_0)\}$. By Claims 9 and 10, $A^h_X \in U_k$. Suppose $X, Y \in T^h \cap C_h$ and $x \in A^h_X \cap A^h_Y$. Then $f(x, X) = f(x, Y) = f(x, X_0) \in K_x$ and $X, Y \in f(x, X_0)$. Since $f(x, X_0) = C$ for some $C \in K_x$, $X \cap Y \in f(x, X_0)$. From the remarks after Claim 8, it follows that $h \frown x \Vdash^* \dot{f}^X_{\zeta_X} | (X \cap Y) = \dot{f}^Y_{\zeta_Y} | (X \cap Y)$. $\square$

Lemma 11. *There is $\bar{S} \subset [\lambda]^{<\nu^+}$ stationary, conditions $\langle p_X : X \in \bar{S}\rangle$ with stem$(p_X) = \bar{h}$ and ordinals $\langle \zeta_X : X \in \bar{S}\rangle$ such that whenever $X, Y \in \bar{S}$, $p_X \wedge p_Y \Vdash \dot{f}^X_{\zeta_X} | (X \cap Y) = \dot{f}^Y_{\zeta_Y} | (X \cap Y)$.*

Proof. We will define a decreasing sequence of clubs $\langle C_k : k \geq n\rangle$ and $A^X_k \in U_k$ for $X \in C_k$, together with the convention $C_{n-1} = [\lambda]^{<\nu^+}$. Each p_X will be of the form $\langle \bar{h}, A^X_n, A^X_{n+1}, ...\rangle$. Assuming that A^X_i has been defined for $n \leq i < k$, for any $h \supset \bar{h}$ with length$(h) = k$, let $T^h := \{X \in T : X \in C_{k-1} \wedge (\forall i \in [n, k)) h(i) \in A^X_i\}$.

Our induction hypothesis will be the following: for every $k \geq n$, if $X, Y \in C_k \cap T$ and $h = \bar{h} \frown \bar{y}$, with $\bar{y} = \langle y_n, ...y_{k-1}\rangle, y_i \in A^X_i \cap A^Y_i$, then T^h is stationary and $\forall z \in A^X_k \cap A^Y_k$ with $h \prec z$, we have $h \frown z \Vdash^* \dot{f}^X_{\zeta_X} | (X \cap Y) = \dot{f}^Y_{\zeta_Y} | (X \cap Y)$.

For $k = n$, $T^{\bar{h}} = T$. By Lemma 4, this satisfies the hypothesis of Lemma 5. Let $A^X_n = u_{\bar{h}}(X)$. By Lemma 5, this is as required. Let $C_n = C_{\bar{h}}$ as from Lemma 5.

Now assume we have done the construction for $n \leq i < k$ and let $h \supset \bar{h}$ have length k. If T^h is nonstationary, let C_h be a club disjoint from T^h. If T^h is stationary, then by the induction assumption, the hypotheses of Lemma 5 are satisfied. Let C_h and $A^h_X := u_h(X)$ be as in the conclusion. Take $C_k = \bigcap_h C_h$ and $A^X_k = \Delta_h A^h_X$, with the intersections taken over all $h \supset \bar{h}$

of length k.

We must verify that the induction hypothesis still holds. Let $X, Y \in C_k \cap T$ and $h = \bar{h} \frown \bar{y}$, with $\bar{y} = \langle y_n, ...y_{k-1}\rangle, y_i \in A_i^X \cap A_i^Y$. First note that $X, Y \in T^h$ by definition. Since $\emptyset \neq C_k \cap T^h \subset C_h \cap T^h$, T^h must be stationary. For any $z \in A_k^X \cap A_k^Y$ with $h \prec z$, $z \in A_X^h \cap A_Y^h$ by definition of diagonal intersection. By Lemma 5, $h \frown z \Vdash^* \dot{f}^X_{\zeta_X} |(X \cap Y) = \dot{f}^Y_{\zeta_Y} |(X \cap Y)$.

Having completed the inductive construction, let $\bar{S} = T \cap \bigcap_k C_k$ and for each $X \in \bar{S}$, let $p_X = \langle \bar{h}, A_n^X, A_{n+1}^X, ...\rangle$. We will next show that for $X, Y \in \bar{S}$ fixed, $D := \{q \leq p_X \wedge p_Y : q \Vdash \dot{f}^X_{\zeta_X} |(X \cap Y) = \dot{f}^Y_{\zeta_Y} |(X \cap Y)\}$ is dense. That will imply, $p_X \wedge p_Y \Vdash \dot{f}^X_{\zeta_X} |(X \cap Y) = \dot{f}^Y_{\zeta_Y} |(X \cap Y)$.

Let $p \leq p_X \wedge p_Y$. Then $p = \langle \bar{h}, y_n, ...y_{k-1}, A_k, A_{k+1}, ...\rangle$ with $y_i \in A_i^X \cap A_i^Y$ for $n \leq i < k$ and $A_i \subset A_i^X \cap A_i^Y$ for $i \geq k$. Let $\bar{y} = \langle y_n, ..., y_{k-1}\rangle$. Since $X, Y \in C_k \cap T$, if we take $z \in A_k$ with $h \prec z$, by our inductive construction, $\bar{h} \frown \bar{y} \frown z \Vdash^* \dot{f}^X_{\zeta_X} |(X \cap Y) = \dot{f}^Y_{\zeta_Y} |(X \cap Y)$. Let q be a witness. After intersecting each A_i^q with A_i for $i > k$, we may assume without loss of generality $q \leq p$. Then $q \in D$. $\square$

We are finally ready to complete the proof that F has an unbounded branch.

Let $B = \{X \in \bar{S} : p_X \in G\}$. We will show that B is stationary. Suppose not. Then there is a club $C' \in V[E][G]$ and $q \in G$ such that $q \Vdash \dot{C}'$ is club $\wedge\ \dot{B} \cap \dot{C}' = \emptyset$. Applying Lemmas 3 and 4 densely below q then strengthening q if necessary, we may assume stem$(q) = \bar{h}$. By Lemma 1, there is a club $C \in V[E]$ such that $q \Vdash \dot{B} \cap C = \emptyset$. Let $X \in \bar{S} \cap C$. Then stem$(p_X) =$ stem(q). Taking r a common extension of p_X and q, we have $r \Vdash \dot{B} \cap C = \emptyset$, which implies $r \Vdash X \notin \dot{B}$. But then $r \Vdash p_X \notin G$, which is impossible.

Let $b = \bigcup\{f^X_{\zeta_X} : X \in B\}$. Then this is an unbounded branch as required, completing the proof.

4. Open Problems

Problem 1. *Can we consistently obtain the strong tree property at κ^+ with κ strong limit and $\neg SCH_\kappa$ for $\kappa = \aleph_{\omega^2}$? How about $\kappa = \aleph_\omega$?*

We may attempt to bring κ down to a small cardinal by adding interleaved collapses to the forcing. Unfortunately, this does not work at $\kappa = \aleph_\omega$ because doing so adds a weak square sequence[10], which implies the failure of the tree property. However, this may work at $\kappa = \aleph_{\omega^2}$.

Problem 2. *Can we consistently obtain the strong tree property at κ^+ and κ^{++} with κ strong limit? If so, can we bring this result down to $\kappa = \aleph_{\omega^2}$? How about $\kappa = \aleph_\omega$?*

The answer to the first two questions is yes for the tree property. See [9] and [11].

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